The Equation of Motion of an Inverted Pendulum

A.K.A "The Balancing Act"

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1. INTRODUCTION

BACKGROUND RESEARCH

I remember walking through the doors of the old Albertsons store in Canyon Park. Every single time I walked through, I would look back and laugh. The automatic doors would swing out, stabilize by swinging side to side, then violently smash themselves together despite their efforts. The sole reason for this was poor control mechanics; the mathematical algorithms by which the door's motors operated on were imperfect in terms of stability. This scenario, combined with my passion for robotics and mathematics, led me to discover the inverted pendulum.

For fifty years, the inverted pendulum has been the benchmark for control theory and robotics. Control theory is an interdisciplinary field of engineering and mathematics that has applications in the behavior of dynamical systems, modified by inputs and feedback. A pendulum is simply a weight hanging on a string or a light rod, which follows the basic rules of oscillatory motion. Examples of pendulums are found in playground swings, grandfather clocks, and more. It is an example of a robust system. Robustness can be defined as the ability of a system to resist change without adapting its initial stable configuration. In other words, the system of a robust pendulum can take multiple inputs and return to its equilibrium without difficulty.
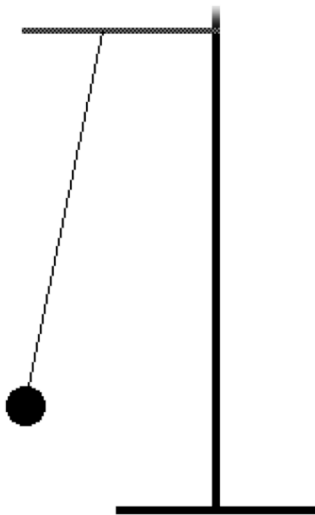
On the other hand, an inverted pendulum is a weight balanced on top of a light rod; it's the same system, turned upside-down. An inverted pendulum is the embodiment of an unstable, not robust system. When given even the smallest inputs, or disturbances, the inverted pendulum isn't able to return to its equilibrium state. It's the definition of a balancing act.

An inverted pendulum, attached to a moving cart, is the standard of inverted-pendulum robots. The cart only moves in one axis, as does the inverted pendulum. The angle of the inverted pendulum is measured using a rotary encoder, and the position of the cart is measured using rotary encoders within motors on the wheels. These data, the angle of the pendulum and the position of the cart, along with

time, are all that's needed to control and stabilize the inverted pendulum by moving the cart using the motors.
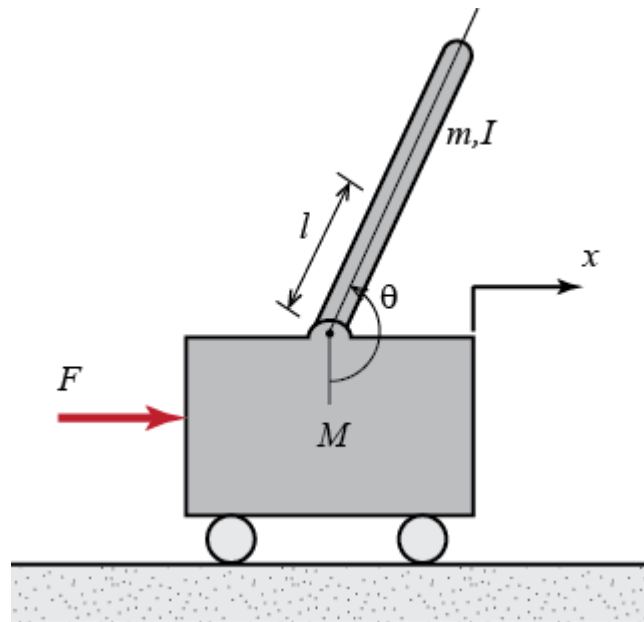

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

These are the dynamic equations of the inverted pendulum, in a state-space model. State-space equations model a system by using a set of differential equations. This equation is written using Newton's notation. The variable x represents the position of the cart, and φ represents the angle of the pendulum with respect to the normal. Each dot above a variable represents differentiation with respect to time. $\dot{x}$ is $\frac{dx}{dt}$ which is velocity, and $\ddot{x}$ is $\frac{d^2x}{dt^2}$, which is acceleration. The same principle applies to φ (Messner).

The control algorithm is the mathematical process, based off the above equations, used to stabilize the robot. I used a PID (Proportional – Integral – Derivative) controller, which follows a closed loop, or feedback, control system. This involves a cyclical process, in which two variables, the desired effect and the current state, are inputted into a PID controller, processed with disturbances, outputs a value to be executed by the system, and the process repeats.
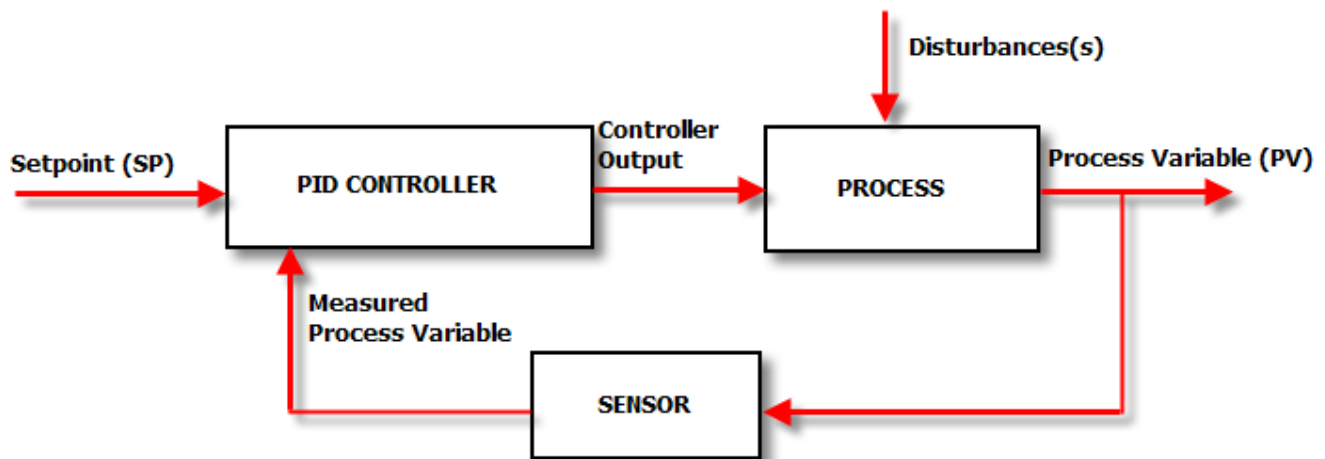
In the case of the inverted pendulum robot, the "setpoint," or desired value, is keeping the inverted pendulum upright, or parallel to the normal/ vertical. The measured process variables are the current angle of the pendulum with respect to the normal and the x-position of the cart (using a rotary encoder sensor), in addition to the first-order and second-order derivatives of those values with respect to other values. The PID controller, in layman's terms, takes "the difference" of the two values, using higher calculus, and outputs a correction. In this case, the correction is an adjustment in the cart using the motors.

RESEARCH QUESTION
What is the equation of motion of an inverted pendulum?

HYPOTHESIS
When given disturbance forces, the inverted pendulum should tip over. If the control algorithms previously listed are properly executed, either through a simulation or in a physical robot, the cart should balance the inverted pendulum, even if there is excess weight at the end or more friction in the joints.

2. METHODOLOGY

OVERVIEW
This experiment involves creating a two-dimensional, digital simulation of an inverted pendulum mounted to a motorized cart. Each simulation begins with the cart and inverted pendulum in a stable position, with the inverted pendulum upright, or 180 degrees to the vertical. The first simulation is open-loop, i.e. the motors don't give any feedback and the cart and pendulum are completely free to move; there is a disturbance force of 100N applied for .01 seconds, and properties such as force on the cart,

position of the cart, and angle of the inverted pendulum are recorded for 10 seconds. The second simulation is closed-loop model, i.e. the motors give feedback and the cart resists the disturbance forces, keeping the inverted pendulum from falling over. Since the feedback is based off the equation of motion of an inverted pendulum, if the second simulation succeeds in balancing the inverted pendulum, the equation will have proven to work.

MATERIALS

- Simulink®: "a block diagram environment for multidomain simulation and Model-Based Design. It supports simulation, automatic code generation, and continuous test and verification of embedded systems." ("Simulink Overview").

- A modern computer

STEPS

1. Create a linearized model by following this Simulink Modeling tutorial:
   http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SimulinkModeling

2. Simulate the model without any external forces. Simultaneously record the disturbance force on the cart (N), position of cart (m), and angle of the inverted pendulum (radians).

3. Simulate the model, for ten seconds, with an external disturbance force of 1000N applied for .01 seconds. Simultaneously record the disturbance force on the cart (N), position of cart (m), and angle of the inverted pendulum (radians).

4. Connect a controller to the cart. This controller should only be able to apply forces to the cart the same way the external forces do.

5. Simulate the model with the controller, for ten seconds, with two disturbance forces similar to the first one. The first push is towards the right (positive) and the second push is to the left (negative). Simultaneously record the disturbance force on the cart (N), position of cart (m), and angle of the inverted pendulum (radians) on one graph. On another graph, simultaneously record net force on the cart (N), the disturbance force applied (N), and the force applied by the controller on the cart.

LABELED DIAGRAM

- M – mass of the cart (0.5 kg)

- m – mass of the pendulum (0.2 kg)

- b – coefficient of friction for cart (0.1 N/m/sec)

- l – length to pendulum center of mass (0.3 m)

- I – mass moment of inertia of the pendulum (0.006 kg $\times m^2$)

- F – force applied to the cart

- x – cart position coordinate

- θ – pendulum angle from vertical (down)

Free-body Diagram:
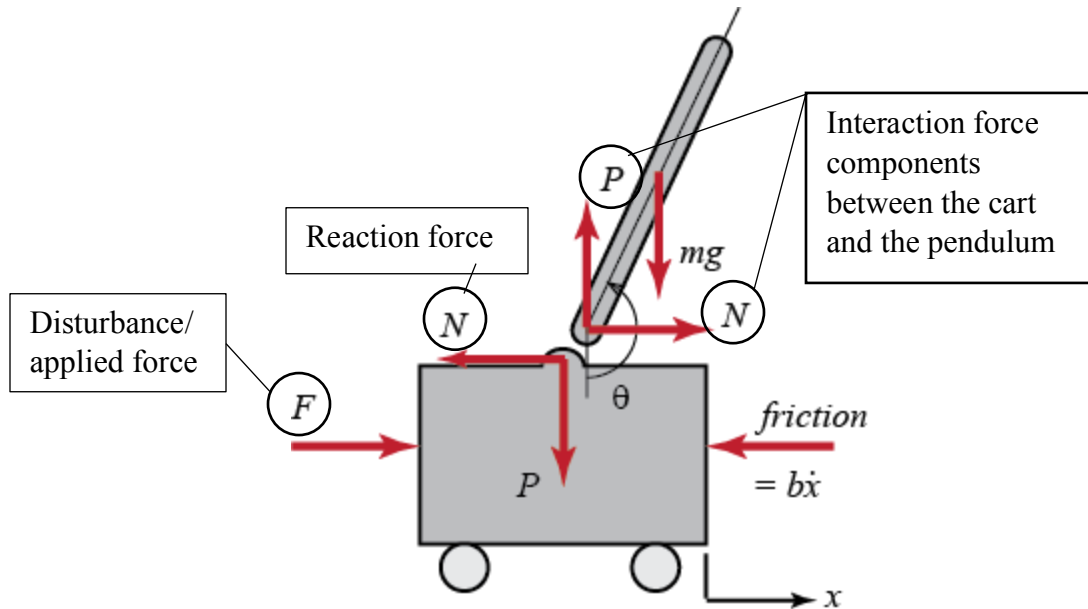
It is necessary to include the interaction forces N and P between the cart and the pendulum, which requires modeling the x- and y- components of the translation of the pendulum's center of mass and its rotational dynamics (Messner). We can express x_p and y_p as exact functions of θ, and their derivatives.

The x-component equations follow as such:

$$x_p = (x + l sin\theta)$$

$$\dot{x}_p = \dot{x} + l(\dot{\theta}cos\theta)$$

$$\ddot{x}_p = \ddot{x} + ((l\dot{\theta})(\dot{\theta}(-sin\theta)) + (cos\theta)(l\ddot{\theta}))$$

$$\ddot{x}_p = \ddot{x} - l\dot{\theta}^2 sin\theta + l\ddot{\theta}cos\theta$$

The y-component equations follow as such:

$$y_p = -lcos\theta$$

$$\dot{y}_p = l\dot{\theta}sin\theta$$

$$\ddot{y}_p = l\dot{\theta}^2 cos\theta + l\ddot{\theta}sin\theta$$

You can substitute the equations into Newton's second law:

$$F = ma$$

$$N = m(\ddot{x_p} = \ddot{x} - l\dot{\theta}^2 \sin\theta + l\dot{\theta}\cos\theta)$$

$$P = m\left(l\dot{\theta}^2 \cos\theta + l\ddot{\theta}\sin\theta\right) + F_g$$

Now, these equations can be represented within Simulink, along with other similar yet more complex equations.

SAFETY CONSIDERATIONS

This experiment solely relies on computer simulation, which has almost no safety, ethical, or environmental issues. If I were to build a hardware model using robotic servos, encoders,

potentiometers, and aluminum parts, there would only be safety issues related to electricity. To be safe, my brother and I have built an electrical engineering lab in the loft. Attached to this lab is a wrist strap connected to the ground to prevent static discharge to electronics.

3. RESULTS AND ANALYSIS

INVERTED PENDULUM – OPEN LOOP MODEL

This graph represents a simulation of the inverted pendulum without any controller attached. The simulation runs for ten seconds, and one set of data is plotted. In the simulation, the inverted pendulum begins upright, in equilibrium. Then, a disturbance force of 1000N is applied for .01 second, as if someone shoved the box.

RAW DATA

Time (sec)	Net Force (N)	Position (m)	Angle (radians)
0	0	0	3.141593

1	0	-9.4E-18	3.141593
2	1000	-3.6E-17	3.141593
2.956	0	1.301533	3.464519
3.956	0	2.510836	4.42302
4.956	0	3.387802	6.806573
5.956	0	4.255956	8.664989
6.956	0	5.062617	9.267003
7.956	0	5.739088	9.508919
8.956	0	6.337424	9.913141
10	0	6.881552	11.33249

Force of 1000N applied for 0.01s (to the right)

Force (N)

time (sec)

Position (meters)

time (sec)

Angle (radians)

The inverted pendulum is upright & balanced at π radians to the vertical

The pendulum is down at any multiple of 2π.

time (sec)

ANALYSIS

For the first two seconds of the simulation, there are no external forces applied to the inverted pendulum, the cart is at position zero, and the angle of the pendulum is at pi relative to the vertical. At two seconds, a force of 1000N is applied for .01 seconds. That is the only external force applied for the duration of the simulation. Because of this, the cart moves in the positive direction, which is right. The pendulum swings counter-clockwise, completing one rotation (at 2pi radians) and a half, stabilizing at the bottom (at 4pi radians).
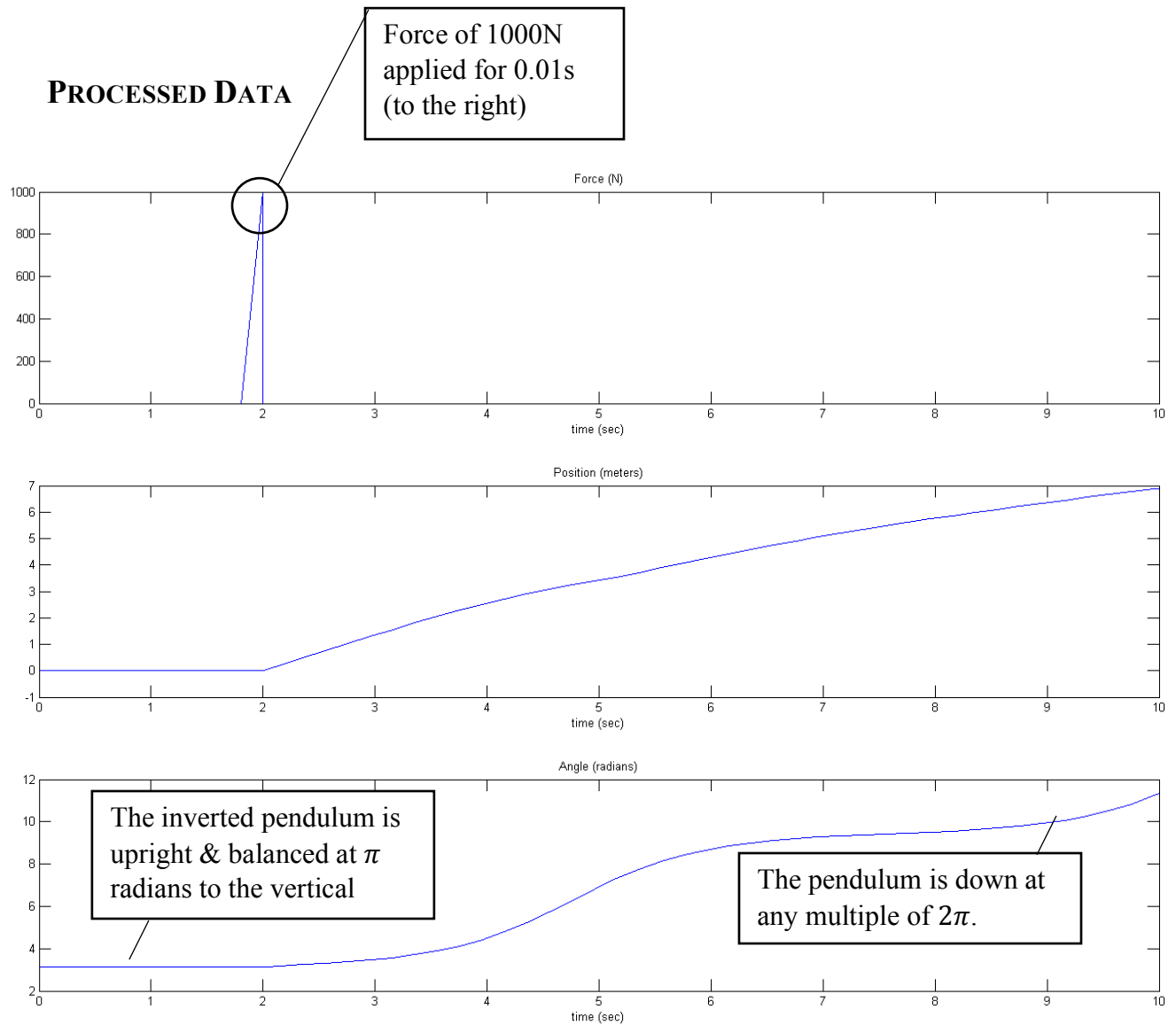
INVERTED PENDULUM AND CONTROLLER – CLOSED LOOP MODELS

This graph represents a simulation of the inverted pendulum that runs with a controller attached. The simulation runs for ten seconds, and two sets of data are plotted. In the simulation, two disturbance forces similar to the first example are applied; the first one pushes to the right, then the second one

pushes to the left. The first plot displays the net force on the cart, along with the pendulum and position of the cart. The second plot displays the net force on the cart, along with the disturbance force on the cart, and the force applied by the controller on the cart.

RAW DATA

time (sec)	Net force (N)	Position (m)	Angle (radians)	Disturbance Force (N)	Controller Force (N)
0	0	0	3.141593	0	0
1	0	-9.4E-18	3.141593	0	0
2	1000	-3.6E-17	3.141593	1000	0
2.12733	-4.65726	0.12642	3.165594	0	-4.65726
2.415882	-0.47342	0.086116	3.161022	0	-0.47342
2.450591	-0.16707	0.068955	3.158399	0	-0.16707
3.013218	0.242765	-0.10012	3.138497	0	0.242765
4.002644	0.003032	-0.18758	3.141727	0	0.003032
5.031011	0.016211	-0.29701	3.141595	0	0.016211
5.976569	-0.00014	-0.39694	3.141591	0	-0.00014
6	-1000.01	-0.39942	3.14159	-1000	-0.00864
6.050767	5.658859	-0.47021	3.129235	0	5.658859
6.469864	-0.02106	-0.50836	3.126247	0	-0.02106
6.999636	-0.31604	-0.40617	3.14473	0	-0.31604
8.022436	-0.05768	-0.4233	3.141472	0	-0.05768
10	-0.01279	-0.42247	3.141593	0	-0.01279

Force of 1000N applied for 0.01s (to the right)

Force of -1000N applied for 0.01s (to the left)

Force (N)

time (sec)

Position (meters)

The controller reacts by moving the cart back-and-forth rapidly

The small scale shows how little the inverted pendulum is affected, demonstrating the accuracy of the controller based off the equation of motion

time (sec)
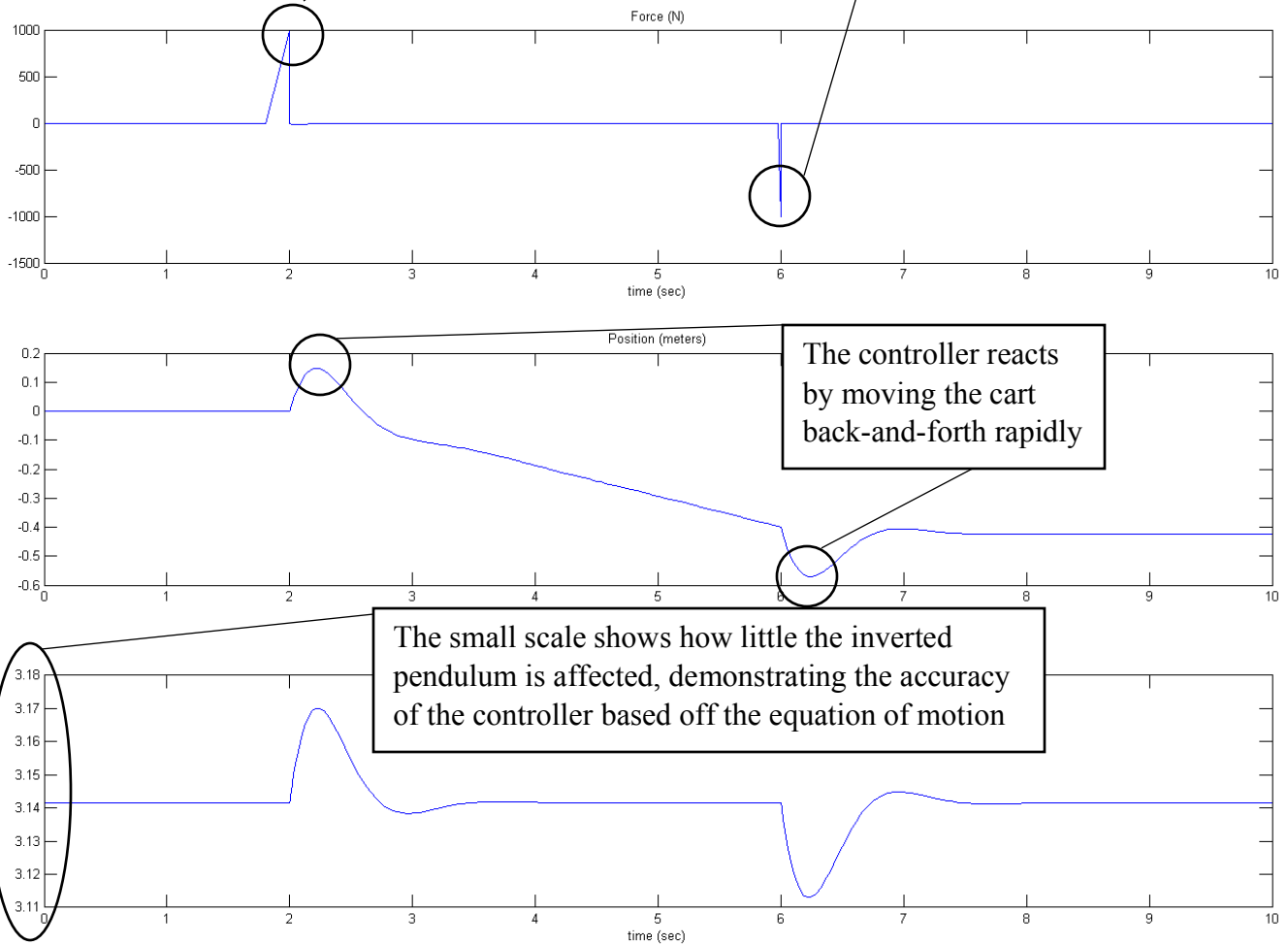
ANALYSIS

Again, for the first two seconds of the simulation, there are no external forces applied to the inverted pendulum, the cart is at position zero, and the angle of the pendulum is at pi relative to the vertical. At two seconds, a force of 1000N is applied for .01 seconds. The initial force causes the cart to move to the right, but the controller quickly counter-acts the force by rapidly jerking to the left then gradually moving to the left to keep the pendulum stable. Meanwhile, the inverted pendulum slightly tilts left then right, but keeps upright.

In addition, at six seconds, there is another force of -1000N applied for .01 seconds (to the left). The cart moves to the left, and the controller reacts by quickly moving to the right and keeping still. The inverted pendulum slightly tilts right then left, but keeps upright.

PROCESSED DATA: GRAPH 2

The net force includes the controller and disturbance forces combined

The controller force opposes the direction of the disturbance forces
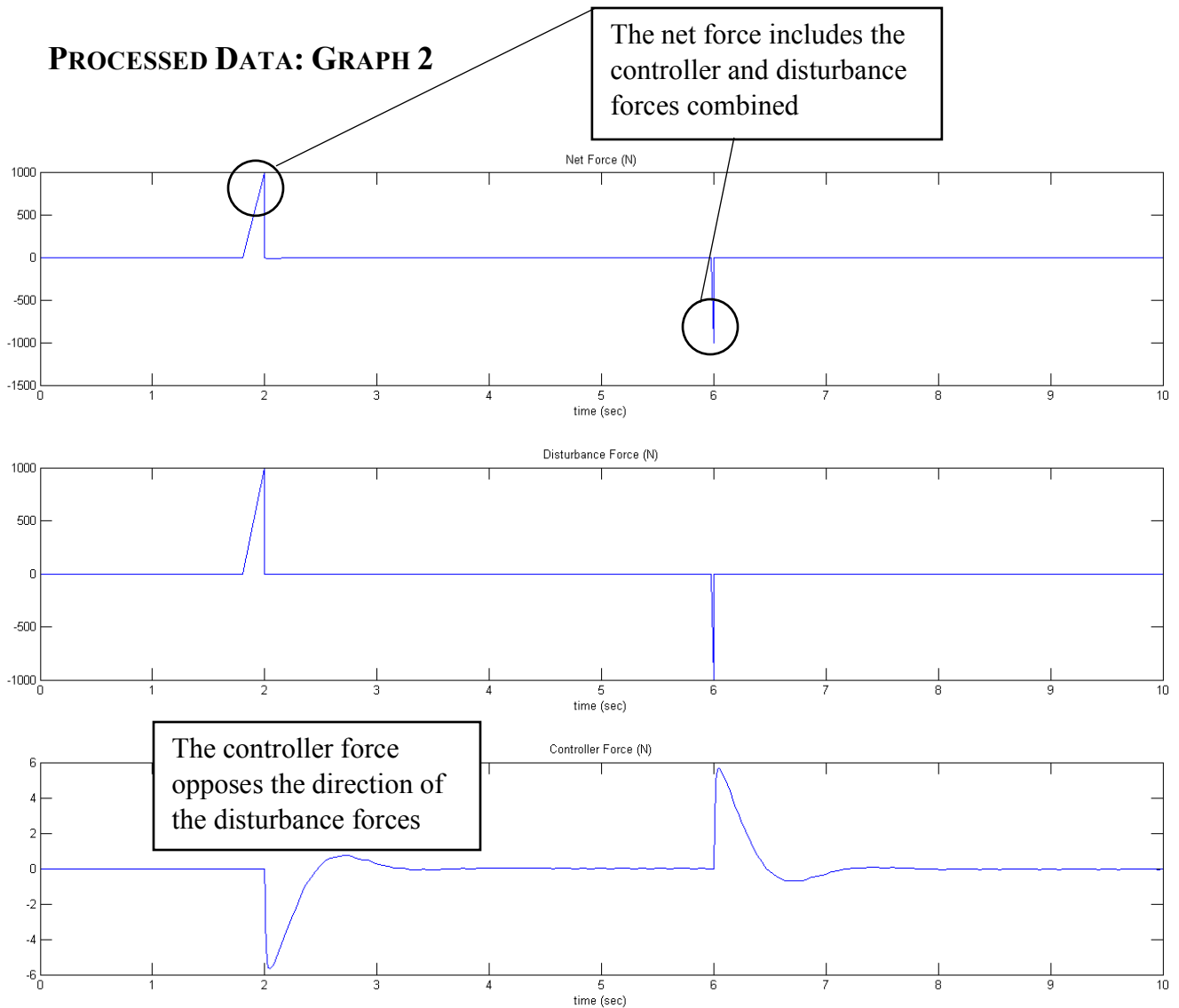
ANALYSIS

The same closed-loop scenario occurs as in the previous graph; this graph simply displays the forces involved. The disturbance force is the same as in the previous graphs. The controller forces occur immediately after or slightly during the time the disturbance forces are applied, and they counteract the force with an initial kickback and a relatively gradual stabilization from the kickback. At two seconds, the controller rapidly moves left, then right, and keeps a small, consistent force. At six seconds, the

controller quickly moves right, then left, and continues to make minute forces to keep the inverted pendulum stable.

UNCERTAINTIES

Although this experiment took place in a simulator, there are uncertainties in the linearized model itself. The simulator is complex, and therefore, isn't perfect. However, the miniscule inaccuracies in this experiment are negligible; this is presented in the accuracy of the data in the tables.

4. CONCLUSION AND EVALUATION

CONCLUSION

Ultimately, the control algorithm functioned properly by stabilizing the inverted pendulum after multiple impacts. The behavior of the cart depended on the algorithm, and the behavior of the algorithm depended on the state-space model, which was the equation of motion of inverted pendulum. Therefore, since the eventual functionality was successful, the underlying principles must be true; the equation of motion of an inverted pendulum is justified.

COMPARISON TO CONTEXT

The state-space model used in this experiment has been accepted for over 50 years. The procedures from this experiment are derived from control tutorials taught by University of Michigan, Carnegie Mellon University, and University of Detroit Mercy in conjunction. Because this experiment was conducted as a simulation, the difference between my results and that of another similar experiment are little to none.

STRENGTHS AND WEAKNESSES

A definite strength of my experiment is that it was simulated using one of the most popular applications today: Matlab. It is a justified means of analyzing the validity of control algorithms. On the other hand, I am not able to visualize the cart unless I had an API for animations. Also, since it was simulated, I did not encounter inconsistencies in measurement that would have occurred if I had built a real pendulum balancer. Encoders, both linear and rotational, have their own inconsistencies that may impact the algorithm. Surprisingly, factors such as mass of the pendulum, mass of the cart, friction between the pendulum and the cart, and friction between the cart and the rail/ ground do not invalidate the experiment. The algorithm, since it only takes the angle of the pendulum as an input, is able to adapt to such changing conditions. Thus, the only weaknesses in building a real pendulum balancer lie in the inaccuracies of the sensors.

IMPROVEMENTS

I am currently implementing a real pendulum balancer (see Appendix B) to visualize this concept in action. Moreover, I could implement different, more adaptive control algorithms, each with their own way of kicking-back and stabilizing, to understand the multifarious forms of such algorithms.

5. WORKS CITED

Works Cited

Messner, Bill, Prof., and Dawn Tilbury, Prof. "Inverted Pendulum: System Modeling." *Control Tutorials for Matlab & Simulink*. Matlab, n.d. Web. 7 Dec. 2015. <http://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling>.

Peacock, Finn. *An Idiot's Guide to the PID Algorithm*. N.p.: n.p., 2008. Print.

"PID Theory Explained." *National Instruments*. National Instruments, 29 Mar. 2011. Web. 7 Dec. 2015. <http://www.ni.com/white-paper/3782/en/#top>.

Sekar, Arul Selvan. Personal interview. 21 Aug. 1988.

"Simulink Overview." *MathWorks*. MathWorks, n.d. Web. 7 Dec. 2015. <http://www.mathworks.com/products/simulink/>.

"What Are State-Space Models?" *MathWorks*. MathWorks, 2015. Web. 7 Dec. 2015. <http://www.mathworks.com/help/ident/ug/what-are-state-space-models.html>.

6. APPENDICES

APPENDIX A – FULL DATA TABLES

OPEN LOOP MODEL

Time (sec)	Net Force (N)	Position (m)	Angle (radians)
0.000	0	0	3.141593
0.200	0	-3.9E-19	3.141593
0.400	0	-1.5E-18	3.141593
0.600	0	-3.5E-18	3.141593
0.800	0	-6.1E-18	3.141593
1.000	0	-9.4E-18	3.141593
1.200	0	-1.3E-17	3.141593

1.400	0	-1.8E-17	3.141593
1.600	0	-2.3E-17	3.141593
1.800	0	-2.9E-17	3.141593
2.000	1000	-3.6E-17	3.141593
2.001	0	0.000726	3.14173
2.006	0	0.007983	3.143099
2.031	0	0.04419	3.149933
2.156	0	0.223291	3.184039
2.356	0	0.503421	3.240368
2.556	0	0.77613	3.302631
2.756	0	1.041981	3.375536
2.956	0	1.301533	3.464519
3.156	0	1.555301	3.57605
3.356	0	1.803648	3.717916
3.556	0	2.046547	3.899345
3.756	0	2.283103	4.130795
3.956	0	2.510836	4.42302
4.156	0	2.72505	4.784974
4.356	0	2.91949	5.220054
4.556	0	3.090112	5.720577
4.756	0	3.241407	6.262384
4.956	0	3.387802	6.806573
5.156	0	3.543877	7.313372
5.356	0	3.714118	7.756768
5.556	0	3.893938	8.127338
5.756	0	4.076313	8.4273
5.956	0	4.255956	8.664989
6.156	0	4.430075	8.850938
6.356	0	4.597668	8.995583
6.556	0	4.758689	9.108189
6.756	0	4.913505	9.196537
6.956	0	5.062617	9.267003
7.156	0	5.206531	9.324808
7.356	0	5.345717	9.374295
7.556	0	5.480608	9.419213
7.756	0	5.611604	9.462978
7.956	0	5.739088	9.508919
8.156	0	5.863441	9.56052
8.356	0	5.985053	9.621673
8.556	0	6.104331	9.696943
8.756	0	6.221675	9.791848
8.956	0	6.337424	9.913141
9.156	0	6.451697	10.06902

9.356	0	6.564062	10.26914
9.556	0	6.672953	10.52412
9.756	0	6.774903	10.84417
9.956	0	6.864242	11.23637
10	0	6.881552	11.33249

CLOSED LOOP MODEL

time (sec)	Net force (N)	Position (m)	Angle (radians)	Disturbance Force (N)	Controller Force (N)
0	0	0	3.141593	0	0
0.2	0	-3.9E-19	3.141593	0	0
0.4	0	-1.5E-18	3.141593	0	0
0.6	0	-3.5E-18	3.141593	0	0
0.8	0	-6.1E-18	3.141593	0	0
1	0	-9.4E-18	3.141593	0	0
1.2	0	-1.3E-17	3.141593	0	0
1.4	0	-1.8E-17	3.141593	0	0
1.6	0	-2.3E-17	3.141593	0	0
1.8	0	-2.9E-17	3.141593	0	0
2	1000	-3.6E-17	3.141593	1000	0
2.001000	-0.27874	0.000726	3.14173	0	-0.27874
2.003374	-1.44442	0.00417	3.142379	0	-1.44442
2.007779	-3.01463	0.010518	3.143577	0	-3.01463
2.014174	-4.37555	0.019571	3.145286	0	-4.37555
2.022669	-5.24443	0.031179	3.147476	0	-5.24443
2.033631	-5.63616	0.045326	3.150148	0	-5.63616
2.047759	-5.68874	0.062079	3.153314	0	-5.68874
2.066296	-5.52751	0.081523	3.156997	0	-5.52751
2.091493	-5.19893	0.103479	3.161175	0	-5.19893
2.12733	-4.65726	0.12642	3.165594	0	-4.65726
2.176086	-3.83725	0.143759	3.169083	0	-3.83725
2.207408	-3.32731	0.147646	3.170019	0	-3.32731
2.238731	-2.81962	0.146772	3.170108	0	-2.81962
2.277092	-2.21216	0.140313	3.169272	0	-2.21216
2.319422	-1.57282	0.12778	3.167416	0	-1.57282
2.356675	-1.07255	0.1133	3.165203	0	-1.07255
2.386845	-0.75597	0.099916	3.163143	0	-0.75597
2.415882	-0.47342	0.086116	3.161022	0	-0.47342
2.450591	-0.16707	0.068955	3.158399	0	-0.16707
2.492228	0.146588	0.048116	3.155255	0	0.146588
2.533096	0.405866	0.028078	3.15229	0	0.405866
2.566229	0.524563	0.012505	3.150037	0	0.524563
2.594662	0.582981	-0.0002	3.148243	0	0.582981

2.625863	0.642311	-0.0133	3.146443	0	0.642311
2.664572	0.692539	-0.02823	3.144473	0	0.692539
2.707376	0.731789	-0.04292	3.142646	0	0.731789
2.744599	0.731328	-0.05413	3.141355	0	0.731328
2.774385	0.666937	-0.0621	3.140512	0	0.666937
2.803134	0.614412	-0.06897	3.139851	0	0.614412
2.837934	0.560239	-0.07627	3.139237	0	0.560239
2.879991	0.502172	-0.08377	3.13874	0	0.502172
2.921081	0.464141	-0.08986	3.138478	0	0.464141
2.953999	0.383526	-0.09401	3.138401	0	0.383526
2.982114	0.303382	-0.0971	3.138413	0	0.303382
3.013218	0.242765	-0.10012	3.138497	0	0.242765
3.052142	0.185077	-0.10343	3.138685	0	0.185077
3.095266	0.153444	-0.10662	3.138972	0	0.153444
3.124815	0.089101	-0.10862	3.139198	0	0.089101
3.154364	0.045492	-0.11052	3.13944	0	0.045492
3.191609	0.012089	-0.11283	3.139755	0	0.012089
3.235125	0.003659	-0.11551	3.140122	0	0.003659
3.26573	-0.03348	-0.11743	3.14037	0	-0.03348
3.296335	-0.05648	-0.11941	3.140604	0	-0.05648
3.333712	-0.066	-0.12193	3.140869	0	-0.066
3.376045	-0.05167	-0.12495	3.141135	0	-0.05167
3.414348	-0.03121	-0.12786	3.141341	0	-0.03121
3.445213	-0.0616	-0.13034	3.141482	0	-0.0616
3.473945	-0.07838	-0.13274	3.141595	0	-0.07838
3.507649	-0.07693	-0.13568	3.141704	0	-0.07693
3.548618	-0.05936	-0.13941	3.141807	0	-0.05936
3.590189	-0.01685	-0.14337	3.141881	0	-0.01685
3.624379	-0.01789	-0.14674	3.141919	0	-0.01789
3.653034	-0.04265	-0.14964	3.141938	0	-0.04265
3.683454	-0.0447	-0.15278	3.141947	0	-0.0447
3.721122	-0.03343	-0.15674	3.141946	0	-0.03343
3.763956	8.44E-05	-0.16132	3.141931	0	8.44E-05
3.794462	-0.01415	-0.16463	3.141912	0	-0.01415
3.824967	-0.01797	-0.16796	3.141889	0	-0.01797
3.862203	-0.0095	-0.17206	3.141857	0	-0.0095
3.904527	0.018337	-0.17672	3.141819	0	0.018337
3.94298	0.047046	-0.18097	3.141783	0	0.047046
3.97395	0.019829	-0.1844	3.141753	0	0.019829
4.002644	0.003032	-0.18758	3.141727	0	0.003032
4.036205	0.002597	-0.19128	3.141698	0	0.002597
4.077071	0.015207	-0.19577	3.141667	0	0.015207
4.118736	0.050928	-0.20032	3.141639	0	0.050928

4.153082	0.044672	-0.20406	3.141619	0	0.044672
4.181776	0.013557	-0.20718	3.141603	0	0.013557
4.212089	0.004659	-0.21046	3.14159	0	0.004659
4.249605	0.007566	-0.2145	3.141577	0	0.007566
4.292432	0.031564	-0.21909	3.141566	0	0.031564
4.323047	0.012022	-0.22236	3.14156	0	0.012022
4.353663	0.002729	-0.22563	3.141555	0	0.002729
4.390899	0.005015	-0.22958	3.141552	0	0.005015
4.433086	0.026369	-0.23405	3.141552	0	0.026369
4.471437	0.049314	-0.23809	3.141553	0	0.049314
4.50245	0.01973	-0.24137	3.141554	0	0.01973
4.531258	0.000873	-0.24441	3.141555	0	0.000873
4.564891	-0.00183	-0.24795	3.141558	0	-0.00183
4.605704	0.0087	-0.25224	3.141562	0	0.0087
4.647235	0.042242	-0.2566	3.141568	0	0.042242
4.68154	0.035052	-0.26021	3.141571	0	0.035052
4.710305	0.004409	-0.26324	3.141573	0	0.004409
4.740719	-0.00435	-0.26644	3.141576	0	-0.00435
4.77827	-0.00111	-0.27039	3.14158	0	-0.00111
4.821002	0.02337	-0.27488	3.141584	0	0.02337
4.851621	0.004362	-0.2781	3.141586	0	0.004362
4.88224	-0.0043	-0.28133	3.141588	0	-0.0043
4.919457	-0.00118	-0.28525	3.14159	0	-0.00118
4.961621	0.021072	-0.28969	3.141593	0	0.021072
4.999978	0.044897	-0.29373	3.141595	0	0.044897
5.031011	0.016211	-0.29701	3.141595	0	0.016211
5.059832	-0.00201	-0.30006	3.141595	0	-0.00201
5.093456	-0.00404	-0.30361	3.141595	0	-0.00404
5.134247	0.007192	-0.30791	3.141596	0	0.007192
5.17577	0.041258	-0.31229	3.141597	0	0.041258
5.21009	0.034673	-0.31592	3.141597	0	0.034673
5.238872	0.004416	-0.31897	3.141596	0	0.004416
5.269287	-0.00407	-0.32218	3.141595	0	-0.00407
5.306822	-0.00057	-0.32615	3.141595	0	-0.00057
5.349538	0.024058	-0.33066	3.141595	0	0.024058
5.380174	0.005259	-0.33391	3.141594	0	0.005259
5.410809	-0.00331	-0.33715	3.141593	0	-0.00331
5.448022	-0.00015	-0.34108	3.141593	0	-0.00015
5.490163	0.022043	-0.34553	3.141593	0	0.022043
5.528506	0.045677	-0.34958	3.141594	0	0.045677
5.559549	0.017083	-0.35287	3.141593	0	0.017083
5.588388	-0.00114	-0.35592	3.141592	0	-0.00114
5.622021	-0.00324	-0.35947	3.141591	0	-0.00324

5.662801	0.007883	-0.36378	3.141592	0	0.007883
5.704303	0.041708	-0.36816	3.141592	0	0.041708
5.73862	0.035018	-0.37179	3.141592	0	0.035018
5.767415	0.004802	-0.37484	3.141591	0	0.004802
5.797844	-0.00374	-0.37806	3.141591	0	-0.00374
5.835382	-0.00031	-0.38202	3.141591	0	-0.00031
5.878082	0.024201	-0.38653	3.141591	0	0.024201
5.90872	0.005361	-0.38977	3.141591	0	0.005361
5.939359	-0.00326	-0.39301	3.141591	0	-0.00326
5.976569	-0.00014	-0.39694	3.141591	0	-0.00014
6	-1000.01	-0.39942	3.14159	-1000	-0.00864
6.001	0.269925	-0.40025	3.141453	0	0.269925
6.006	2.449772	-0.40801	3.140088	0	2.449772
6.019289	4.969051	-0.42808	3.136566	0	4.969051
6.033104	5.613006	-0.44758	3.133159	0	5.613006
6.050767	5.658859	-0.47021	3.129235	0	5.658859
6.073106	5.433015	-0.4951	3.124965	0	5.433015
6.10307	5.01853	-0.52222	3.120393	0	5.01853
6.143832	4.374648	-0.54858	3.116104	0	4.374648
6.190121	3.558037	-0.56565	3.113566	0	3.558037
6.217672	3.143418	-0.57026	3.113049	0	3.143418
6.245224	2.706834	-0.57139	3.113149	0	2.706834
6.282635	2.122992	-0.56823	3.114105	0	2.122992
6.328654	1.438029	-0.55858	3.116273	0	1.438029
6.358431	1.081189	-0.54983	3.118096	0	1.081189
6.388208	0.748117	-0.53971	3.120138	0	0.748117
6.426055	0.366202	-0.52557	3.122925	0	0.366202
6.469864	-0.02106	-0.50836	3.126247	0	-0.02106
6.499957	-0.19814	-0.49651	3.128502	0	-0.19814
6.530049	-0.35127	-0.48495	3.13068	0	-0.35127
6.567472	-0.50859	-0.4713	3.133228	0	-0.50859
6.610487	-0.65206	-0.45695	3.135876	0	-0.65206
6.641112	-0.68431	-0.44778	3.137552	0	-0.68431
6.671736	-0.70276	-0.43956	3.139043	0	-0.70276
6.709012	-0.71133	-0.43087	3.140601	0	-0.71133
6.751227	-0.71047	-0.42279	3.142031	0	-0.71047
6.789554	-0.69233	-0.41701	3.143037	0	-0.69233
6.820527	-0.61882	-0.41335	3.143663	0	-0.61882
6.849324	-0.55347	-0.4107	3.144106	0	-0.55347
6.882985	-0.49166	-0.40844	3.14447	0	-0.49166
6.923841	-0.42716	-0.40677	3.144717	0	-0.42716
6.965371	-0.38445	-0.4061	3.144785	0	-0.38445
6.999636	-0.31604	-0.40617	3.14473	0	-0.31604

7.028373	-0.23695	-0.40659	3.144623	0	-0.23695
7.058799	-0.18033	-0.40732	3.144459	0	-0.18033
7.09639	-0.12993	-0.40854	3.144204	0	-0.12993
7.139146	-0.10163	-0.41021	3.143869	0	-0.10163
7.169729	-0.05001	-0.41151	3.143614	0	-0.05001
7.200312	-0.01349	-0.41285	3.143354	0	-0.01349
7.237532	0.01125	-0.41449	3.143042	0	0.01125
7.279746	0.012942	-0.41628	3.142704	0	0.012942
7.318134	0.004351	-0.41781	3.14242	0	0.004351
7.349149	0.041999	-0.41893	3.142213	0	0.041999
7.37793	0.065615	-0.41988	3.142039	0	0.065615
7.411534	0.070856	-0.42088	3.141858	0	0.070856
7.452346	0.059886	-0.42191	3.141672	0	0.059886
7.493914	0.022579	-0.42277	3.14152	0	0.022579
7.528244	0.024795	-0.42333	3.141422	0	0.024795
7.556999	0.050723	-0.42369	3.141359	0	0.050723
7.587381	0.053748	-0.424	3.141308	0	0.053748
7.624909	0.042841	-0.42427	3.141265	0	0.042841
7.667659	0.009348	-0.42444	3.141239	0	0.009348
7.69829	0.021857	-0.42449	3.141235	0	0.021857
7.72892	0.024305	-0.42448	3.141239	0	0.024305
7.76614	0.014091	-0.42442	3.141254	0	0.014091
7.808294	-0.01542	-0.4243	3.141281	0	-0.01542
7.846638	-0.04494	-0.42415	3.141311	0	-0.04494
7.877673	-0.02042	-0.424	3.14134	0	-0.02042
7.906504	-0.00557	-0.42386	3.141367	0	-0.00557
7.940138	-0.00685	-0.42369	3.141399	0	-0.00685
7.980926	-0.02128	-0.42349	3.141437	0	-0.02128
8.022436	-0.05768	-0.4233	3.141472	0	-0.05768
8.056748	-0.05243	-0.42314	3.141501	0	-0.05243
8.085535	-0.02305	-0.42302	3.141523	0	-0.02305
8.115961	-0.01512	-0.4229	3.141544	0	-0.01512
8.153503	-0.01891	-0.42278	3.141566	0	-0.01891
8.196212	-0.04341	-0.42266	3.141586	0	-0.04341
8.226845	-0.02428	-0.42259	3.141599	0	-0.02428
8.257477	-0.01525	-0.42253	3.141609	0	-0.01525
8.294689	-0.01771	-0.42247	3.141619	0	-0.01771
8.336832	-0.03898	-0.42243	3.141625	0	-0.03898
8.375178	-0.06174	-0.4224	3.141629	0	-0.06174
8.406222	-0.0324	-0.42238	3.141631	0	-0.0324
8.435059	-0.0135	-0.42237	3.141632	0	-0.0135
8.468689	-0.01063	-0.42237	3.141632	0	-0.01063
8.509468	-0.0209	-0.42238	3.141629	0	-0.0209

8.550974	-0.05395	-0.42239	3.141625	0	-0.05395
8.585293	-0.0467	-0.42241	3.141622	0	-0.0467
8.614086	-0.01605	-0.42241	3.14162	0	-0.01605
8.644514	-0.00711	-0.42243	3.141617	0	-0.00711
8.682049	-0.01014	-0.42244	3.141613	0	-0.01014
8.72475	-0.03429	-0.42247	3.141608	0	-0.03429
8.75539	-0.01526	-0.42248	3.141606	0	-0.01526
8.78603	-0.0065	-0.42249	3.141603	0	-0.0065
8.82324	-0.00949	-0.42251	3.1416	0	-0.00949
8.865373	-0.03154	-0.42252	3.141596	0	-0.03154
8.903713	-0.05509	-0.42254	3.141593	0	-0.05509
8.934761	-0.02655	-0.42254	3.141592	0	-0.02655
8.963606	-0.00837	-0.42254	3.141592	0	-0.00837
8.99724	-0.00631	-0.42255	3.14159	0	-0.00631
9.038014	-0.01752	-0.42255	3.141589	0	-0.01752
9.079511	-0.0514	-0.42256	3.141587	0	-0.0514
9.113828	-0.04481	-0.42256	3.141587	0	-0.04481
9.142628	-0.01472	-0.42255	3.141588	0	-0.01472
9.173061	-0.00627	-0.42255	3.141588	0	-0.00627
9.210597	-0.0098	-0.42255	3.141588	0	-0.0098
9.253292	-0.0344	-0.42255	3.141587	0	-0.0344
9.283933	-0.01565	-0.42254	3.141588	0	-0.01565
9.314574	-0.0071	-0.42254	3.141589	0	-0.0071
9.351783	-0.01029	-0.42253	3.141589	0	-0.01029
9.393912	-0.03246	-0.42253	3.141589	0	-0.03246
9.432252	-0.05606	-0.42253	3.141589	0	-0.05606
9.463302	-0.02757	-0.42252	3.14159	0	-0.02757
9.492149	-0.00938	-0.42251	3.141591	0	-0.00938
9.525783	-0.00729	-0.42251	3.141591	0	-0.00729
9.566555	-0.01842	-0.42251	3.141591	0	-0.01842
9.60805	-0.05219	-0.42251	3.141591	0	-0.05219
9.642368	-0.04553	-0.4225	3.141591	0	-0.04553
9.67117	-0.01537	-0.42249	3.141592	0	-0.01537
9.701604	-0.00684	-0.42249	3.141593	0	-0.00684
9.73914	-0.01027	-0.42249	3.141593	0	-0.01027
9.781832	-0.03475	-0.42249	3.141592	0	-0.03475
9.812474	-0.01594	-0.42248	3.141593	0	-0.01594
9.843116	-0.00732	-0.42248	3.141593	0	-0.00732
9.880325	-0.01044	-0.42247	3.141593	0	-0.01044
9.922452	-0.03253	-0.42248	3.141592	0	-0.03253
9.960791	-0.05607	-0.42248	3.141591	0	-0.05607
9.991842	-0.02755	-0.42247	3.141592	0	-0.02755
10	-0.01279	-0.42247	3.141593	0	-0.01279

APPENDIX B – PHYSICAL INVERTED PENDULUM ROBOT
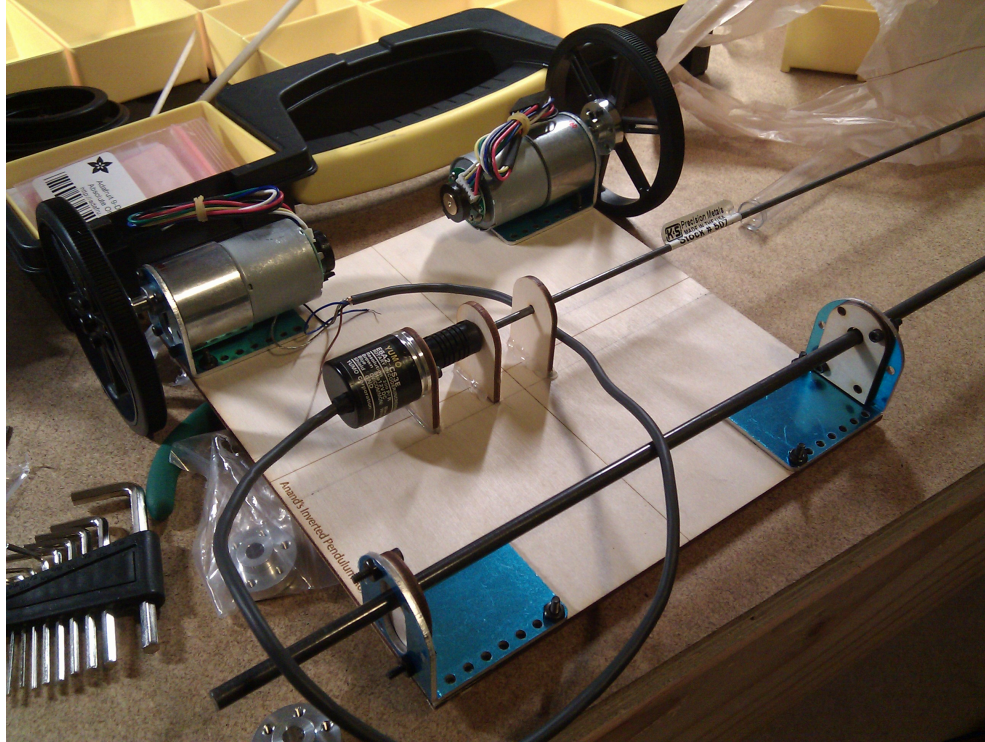
The development of this robot is going relatively quickly. My goal is to finish it within one month. The picture above is of the prototype. The robot is very well designed, with a low center of gravity. With the right materials and measurements, it should have low friction and low error. The entire chassis, pendulum mounts, and several other parts are completely laser cut, so human error in construction is little to none. Once I'm finished, I hope to aggregate the documentation into a research report and post the designs and instructions online.